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1988 J. Phys. A: Math. Gen. 21 L107

(http://iopscience.iop.org/0305-4470/21/2/008)

J. Phys. A: Math. Gen. **21** (1988) L107-L112. Printed in the UK

LETTER TO THE EDITOR

Finite-size electrical resistivity and resistance in fractals†

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Received 21 August 1987, in final form 4 November 1987

Abstract. It is shown that, from an ansatz recently proposed by Dekeyser *et al* for anomalous diffusion and electrical resistance in fractals, one can derive predictions for the relation between intensive (resistivity) and extensive (resistance) quantities in these structures, which differ from the usually assumed forms. Numerical predictions for the ratio of finite-size resistance to resistivity in percolation clusters in space dimensions $2 \leqslant d \leqslant 6$ are made, which are amenable to testing via, e.g., Monte Carlo simulations.

The transport properties of fractals, notably diffusion and electrical conductivity (or resistivity) have attracted considerable interest recently (see, e.g., Pietronero and Tosatti 1986 and references therein). As regards electrical current-conduction aspects, two physical quantities are usually studied: the resistance Ω of a sample (an extensive parameter) and the resistivity ρ (or its inverse, the conductivity σ), which is a local property. For homogeneous, non-fractal substances $\sigma^{-1} = \rho = \text{constant}$ (independent of sample size), and elementary series-parallel arguments give for the resistance $\Omega(R)$ of an R^d hypercube:

$$\Omega(R) = \sigma^{-1} R^{2-d}. \tag{1}$$

Close to the critical probability p_c, the incipient infinite percolation cluster displays a fractal structure when viewed in length scales R such that lattice spacing $\ll R \ll$ correlation length $\zeta(p) \sim |p - p_c|^{-\nu}$. The exponents ζ and t are defined by:

$$\Omega(p) \sim |p - p_c|^{-\zeta} \tag{2a}$$

$$\sigma(p) \sim |p - p_c|^{t} \tag{2b}$$

(see, e.g., Fisch and Harris 1978).

Finite-size scaling then gives, for $R \leqslant \xi$,

$$\Omega(R) \sim R^{\zeta/\nu} \tag{3a}$$

$$\sigma(R) \sim R^{-t/\nu}. \tag{3b}$$

Note that conductivity, which is in principle a local property, now depends on sample size. This reflects the fractal's lack of translational symmetry. For further comments on scale-dependent 'local' properties such as conductivity or diffusion coefficient, and a discussion of crossover between 'anomalous' and 'classical' regions, see Gefen *et al* (1983).

† Work partially supported by FINEP, CAPES and CNPq.

In order to relate the exponents ζ and t, assumptions must be made on the structural relationship between the (local) conductivity and the electrical resistance of a finite-size fractal sample. By assuming a simple 'nodes and links' model for the percolation cluster, de Gennes (1976) proposed that

$$\zeta = t + (2-d)\nu \tag{4}$$

which essentially amounts to assuming that, in the fractal, the combined resistance of a finite sample can be obtained from the local resistivity in the same way as in a compact (Euclidean) structure (see (1) above). An equivalent assumption is made also for non-random fractals such as the Sierpinski gasket (Gefen *et al* 1984).

In this letter, we show that it is possible to extract an alternative expression relating ζ and t, from an ansatz recently proposed by Dekeyser *et al* (1987) for anomalous diffusion and electrical resistance in fractals. We calculate the amount by which such an expression differs from (4) above; we then analyse available data ζ and t. We show that, although existing data are not incompatible with the existence of corrections such as those predicted here, it seems that a final conclusion cannot be drawn as yet. We think that accurate calculations of ζ and t separately would be welcome, in order to establish the precise way in which these quantities relate to each other, thus providing a clearer understanding of the dynamics of fractals.

Dekeyser *et al* (1987) assume that current is carried in a fractal along paths which are self-avoiding walks (SAW). Assuming that the number of steps in a SAW on a fractal cluster scales with its end-to-end distance as

$$N_{\mathrm{SAW}} \sim R^{d_{\mathrm{SAW}}}(\ln R)^{b} \tag{5}$$

and that the average end-to-end distance R_{RW} of a random walk (RW) on a fractal cluster of Hausdorff dimension $\bar{d}$ varies with the number of steps N as

$$R_{\mathrm{RW}}^{2} \sim N^{2/d_{\mathrm{w}}}(\ln N)^{2\alpha}. \tag{6}$$

Dekeyser *et al* (1987) give a Flory-like argument, in which energy scales as $N_{\mathrm{SAW}}^{2}/R^{\bar{d}}$, and entropy as $R^{2}/R_{\mathrm{RW}}^{2}$. From the balance of the two terms it is easy to obtain

$$d_{\mathrm{SAW}} = \tfrac{1}{2}(2+\bar{d})/(1+d_{\mathrm{w}}^{-1}) \tag{7a}$$

$$b = -\alpha/(1+d_{\mathrm{w}}^{-1}). \tag{7b}$$

Equation (7*a*) is equation (7) of Dekeyser *et al* (1987); equation (7*b*) is not quoted there. If the fractal is itself a two-dimensional RW cluster (which was those authors' main concern), the existence of logarithmic corrections to the number of distinct visited sites (Montroll and Weiss 1965) is a plausibility argument for $\alpha \neq 0$ (then $b \neq 0$, by (7*b*)). In the case treated here, that of percolation clusters, logarithmic corrections are not to be expected, thus we assume $\alpha = b = 0$ in what follows. At the end of this work, we shall return briefly to RW clusters.

In the ansatz of Dekeyser *et al*, it is assumed that the current-carrying paths along the fractal are SAW (instead of essentially straight lines, as in homogeneous systems) in parallel; further, it is assumed that essentially the whole fractal takes part in conduction, so the number of parallel paths is proportional to the total number of bonds divided by the number of bonds in a path. Thus, one obtains

$$\Omega(R) \sim N_{\mathrm{SAW}}^{2}(R) R^{-\bar{d}} \tag{8}$$

where $\bar{d}$ is the fractal dimensionality. Since results obtained from (8) are in very good numerical agreement with other estimates for the percolation cluster in $d=2$ (Dekeyser *et al* 1987 and references therein), the ansatz must carry some truth in it, at least for the case of two-dimensional percolation. Those authors actually used the backbone fractal dimensionality $\bar{d}_B$ and the value of d_w^B for RW confined to the backbone in their calculation of N_{SAW} according to (7*a*) above, they also consistently used $\bar{d}_B$ as the fractal dimensionality in (8). Equally accurate results are obtained if, throughout the calculation, one uses the fractal dimension of the full cluster and the random-walk dimension of walks allowed over the full cluster; this reflects a relation between backbone and full percolation cluster which has been discussed in detail by Stanley and Coniglio (1984).

On the other hand, Einstein's relation between conductivity σ, density n and diffusion coefficient $D \equiv \mathrm{d}\langle R_{RW}^2\rangle/\mathrm{d}t$, namely $\sigma = nD$, gives for the conductivity on the scale R of a sample of linear size $R_0 \geqslant R$ at the percolation threshold:

$$\sigma \sim R_0^{\bar{d}-d} R^{2-d_w} \tag{9}$$

(the first factor comes from $n \sim R_0^{-\beta/\nu}$, as given by finite-size scaling; the second comes from the definition of D and $R_{RW}^2 \sim N^{2/d_w}$).

In order to relate $\Omega(R)$ and σ, we now write

$$\Omega(R) \sim \sigma^{-1}(R) R^B. \tag{10}$$

In order to satisfy (8) (with N_{SAW} given by (7*a*)) and (9) (assuming $R_0 \sim R$), one must have

$$B = 2 - d + d_w \frac{(1 + \bar{d} - d_w)}{d_w + 1}. \tag{11}$$

Thus, the ansatz of Dekeyser *et al* (1987) gives

$$\frac{\zeta}{\nu} = \frac{t}{\nu} + 2 - d + d_w \frac{(1 + \bar{d} - d_w)}{d_w + 1} \tag{12}$$

differing from (4) by the last term on the right-hand side. In order to calculate this term, one must be careful: most papers in which an estimate for d_w is reported are studies of the relationship between dynamical (e.g., t) and static (e.g., ν, β) exponents in which assumptions equivalent to (4) above are made (Alexander and Orbach 1982, Rammal and Toulouse 1983, Daoud 1983, Stanley and Coniglio 1984). This may introduce an undesired bias in our results. In order to circumvent this problem, we use estimates of d_w obtained without specific assumptions about the relation between static and conductivity (or conductance) exponents. We refer to the extensive Monte Carlo simulations of Rammal *et al* (1984) from which the spectral dimension $\tilde{d}$ of percolation clusters in Euclidean dimensions $2 \leqslant d \leqslant 6$ is directly extracted, with the result $1.322 \leqslant \tilde{d} \leqslant 1.332$. Thus, together with the relation $d_w = 2\bar{d}/\tilde{d}$, which is obtained under very general assumptions on the structure of diffusion and wave equations (Rammal and Toulouse 1983), gives the desired bias-free estimates of d_w used in table 1.

Physically, the absolute values of the estimates given in table 1 are expected to be upper bounds for the actual corrections. To see this, recall that $B = 2 - d$ would correspond roughly to the current propagating along 'straight' paths on a compact structure, while the ansatz of Dekeyser *et al* assumes propagation alongs SAW on the

Table 1. Corrections given by equation (11) for percolation clusters. Error bars are related only to the uncertainty in $\bar{d}$ as calculated by Rammal *et al* (1984) (see text). In $d=2$, the (presumably exact) $\bar{d}=91/48$ has been used. For $d \geqslant 3$, $\bar{d}$ is as given by Stauffer (1979). In $d=6$, the mean-field values $\bar{d}=4$, $d_w=6$ are assumed.

d	$d_w(1+\bar{d}-d_w)/(d_w+1)$
2	$+0.028 \pm 0.008$
3	-0.22 ± 0.01
4	-0.55 ± 0.02
5	-0.80 ± 0.02
6	$-\frac{6}{7} = -0.857\ldots$

fractal, which are very tortuous (see the discussion in Fisch and Harris (1978), Coniglio (1982)). It is expected that truth must lie somewhere in between these pictures.

Note that corrections are negative for all dimensions except $d=2$; this arises from $d_w > \bar{d}+1$ in (11) for $d \geqslant 3$, which can be traced back to the fact that in higher dimensions 'cutting bonds' (Coniglio 1982) are relatively more important in the structure of the percolation cluster (Gefen *et al* 1981). Thus, trapping within 'blobs' is the physical ingredient responsible for the increasing values of d_w as d grows.

Turning to existing data for comparison, we see that very accurate calculations (to be quoted below) have been performed for the conductivity exponent t/ν in $d=2$ and 3 without use of (4) or similar assumptions. However, as regards the exponent ζ/ν, which has been directly calculated in $2 \leqslant d \leqslant 6$ by Fisch and Harris (1978) by series expansions (see also de Arcangelis *et al* (1985a) for a calculation with similar accuracy), the situation is as follows. In $d=2$ those authors' error bar for ζ/ν (± 0.02) is roughly the size of the corrections predicted in table 1, thus preventing comparison of these corrections against the more accurate values of t/ν reported recently. In $d=3$, as we shall see below, the corrections predicted in table 1 are not inconsistent with the ζ/ν values of Fisch and Harris taken together with recent estimates of t/ν. For $d \geqslant 4$, we know of no calculations of t independent of assumptions similar to (4).

For $d=2$, calculations have been performed for t/ν in order to check the validity of Alexander and Orbach's (1982) conjecture by Zabolitzky (1984), Hermann *et al* (1984) and Lobb and Frank (1984) (the latter authors' 'bulk conductance' is identical to the conductivity). Their results are $t/\nu = 0.973 \pm 0.005$ (Zabolitzky, Lobb and Frank), and $t/\nu = 0.997 \pm 0.010$ (Herrmann *et al*). The result $t/\nu = 0.970 \pm 0.009$ of Hong *et al* (1984) is based on a relation between full cluster exponents and the corresponding backbone exponents derived by Stanley and Coniglio (1984), who assume that (4) holds for *both* sets of exponents. Thus, the identity $d_w^B - \bar{d}_B = d_w - \bar{d}$ used by Hong *et al* is actually *independent* of the assumption of (9); so is their result (except for a corrective term $[d_w(d_w+1)^{-1} - d_w^B(d_w^B+1)^{-1}](1+\bar{d}-d_w) \simeq (1.0 \times 10^{-4})$ predicted by (11)).

It is interesting to note that, for two-dimensional percolation, table 1 and the results for t/ν quoted above imply that the value of ζ/ν must be very close to unity (1.001 ± 0.009, using Zabolitzky's estimate for t/ν). This means that, although the conjecture $t/\nu = 1$ (Daoud 1983) certainly does not hold, $\zeta/\nu = 1$ is a possibly exact result for two-dimensional percolation. This in turn would imply that the breakdown voltage of a random-fuse network at p_c scales linearly with its linear size in $d=2$, contrary to a deduction by de Arcangelis *et al* (1985b), and consistent with the findings of Duxbury *et al* (1987) and of Kahng *et al* (1987) on this problem. Both these latter authors obtain linear scaling of breakdown voltage against size, plus logarithmic corrections.

In three dimensions, Derrida *et al* (1983) report $t/\nu = 2.2 \pm 0.1$ from a transfer-matrix calculation, which is presumably one of the most precise techniques available. Fisch and Harris (1978) report $\zeta = 1.12 \pm 0.02$, which together with what seems to be the most accurate value for ν in 3D, namely $\nu = 0.88 \pm 0.01$ (Heermann and Stauffer 1981) gives $\zeta/\nu = 1.26 \pm 0.03$. While (4) above apparently takes good care of these results, it is worth pointing out that Derrida *et al* (1983) warn of possibly large errors arising from the unknown value of the correction-to-scaling exponent ω. Although their central estimate $t/\nu = 2.2$ comes from a best fit with $\omega = 0.9$, those authors remark that correlation is not significantly worse over the range $0.5 \leqslant \omega \leqslant 1.5$, for which the extrapolated exponent t/ν varies between 2.5 and 2.05 (see figure 3 of Derrida *et al* 1983). They conclude by stating that, should later research fix ω sufficiently accurately, a revised estimate of t/ν would be obtainable from their graph. With the correction $\simeq -0.22$ from our table 1, using ζ/ν as given by Fisch and Harris (1978), we obtain a central estimate of $t/\nu = 2.48$ (corresponding to $\omega \simeq 0.55$), still within the region considered plausible by Derrida *et al* (1983). Of course, the present remarks do not purport to be conclusive; we feel that the existence of corrections such as those predicted here still has to be definitely proved (or else disproved by positive arguments).

Finally, we briefly remark on the extension of the above arguments to the problem treated by Dekeyser *et al* (1987), namely that of diffusion on a two-dimensional RW cluster. Those authors find, from Monte Carlo data and series analysis, that the RMS distance covered by a diffusing particle on a two-dimensional RW cluster varies with time as $\langle R\rangle^{1/2} \sim t^{1/d}w(\ln t)^{\alpha}$, where the numerical values of d_w and α are consistent with the conjecture $d_w = 1/\alpha = 3$. This agrees with the results of Christou and Stinchcombe (1986), which give d_w slightly larger than $\frac{8}{3}$, but is contrary to the work Havlin *et al* (1984) and of Helman *et al* (1984), where it is claimed that diffusion on a 2D RW cluster should be normal because it is space filling, that is, $d_w = 2$ and $\alpha = 0$. Of course, the latter must eventually become true for long enough walks; however, for intermediate length scales, fractal behaviour appears, crossing over towards the classical region later on (see, e.g., Gefen *et al* (1983) for a similar discussion). From their Flory argument, Dekeyser *et al* reobtain $d_w = 3$, but they are unable to reproduce $\alpha = \frac{1}{3}$. This was because they assumed that $\Omega(R) \sim \sigma^{-1}(R) \ln R$ for the relation between resistance and conductance on a 2D RW cluster. Following along the lines of the present work, we find it more natural to *assume* in the case $\Omega(R) \sim \sigma^{-1}(R)(\ln R)^C$ instead of (10), where C is a free parameter to be extracted from the ansatz (note that for 2D RW, $d_w = 3 = \bar{d} + 1$, so (11) above gives $B = 0$ and is then consistent with logarithmic corrections). Proceeding similarly to the argument for percolation clusters, we find $C = \alpha/2$. We interpret this as signalling that, if $\alpha = \frac{1}{3}$ as given by the Monte Carlo and series analysis of Dekeyser *et al* (1987), then $C = \frac{1}{6}$ and vice versa.

In summary, we have shown that, from the ansatz of Dekeyser *et al* (1987) it is possible to derive predictions for the relation between intensive (resistivity) and extensive (electrical resistance) quantities in fractals, which differ from the usually assumed forms. From this, we have extracted numerical predictions for percolation clusters in $2 \leqslant d \leqslant 6$, which can be tested by, e.g., Monte Carlo simulations. Brief comments have been made on the application of some ideas discussed here to diffusion in 2D RW clusters. It is hoped that the ideas discussed here will contribute towards a better understanding of the dynamics of fractals.

The author thanks A L Stella for an illuminating discussion, and N S Branco and R R dos Santos for continued interest and useful remarks along the course of this work. The referee is to be thanked for suggestions towards clarification of presentation, and for drawing the author's attention to relevant references.

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